JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **170**, 322–342 (1992)

The Four-Block Adamjan–Arov–Krein Problem

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Submitted by A. Schumitzky

Received December 14, 1990

This paper is concerned with the mixed sensitivity H^∞ design in the most general, i.e., the four-block, case. This problem involves in a crucial manner the so-called four-block operator Γ, the norm of which is the achievable feedback tolerance. Our objective in this paper is to provide an interpretation for all singular values of Γ. These singular values are given an Adamjan–Arov–Krein interpretation in terms of the L^∞ distance of an L^∞ function to $H^\infty(l)$. Intuitively, the singular values of Γ are the various tolerance levels that can be achieved if we allow a various number of unstable poles in the closed loop. We finally provide an upper bound on the number of singular values. © 1992 Academic Press, Inc.

INTRODUCTION

The present paper is concerned with the feedback diagram of Fig. 1. The objective is to make the closed-loop mapping from the disturbance input w to the error output z, T_{zw}, as small as possible by means of a stabilizing compensator K. The size of the closed-loop transfer function $T_{zw}(j\omega)$ is measured in the "worst case," i.e.,

$$\|T_{zw}\|_\infty := \operatorname*{ess\,sup}_{\omega \in \mathbb{R}} \lambda_{\max}^{1/2}(T_{zw}^T(-j\omega)\, T_{zw}(j\omega)).$$

* Supported by NSF Grants DMS-8701615 and DMS-9101400.

† Supported in part by NSF Grants DMS-8703954, ECS-9113088 and in part by AFOSR Grant 89-0398.

0022-247X/92 $5.00

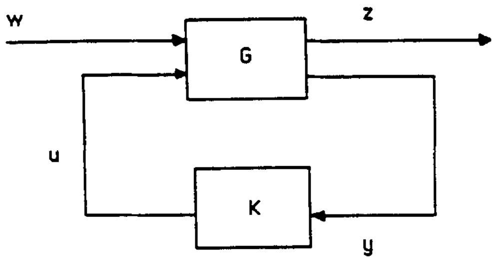

FIGURE 1

Therefore, the control objective is the well-known "worst case," or H^∞ design criterion:

$$\inf_{K \text{ stabilizing}} \|T_{zw}\|_\infty = \mu.$$

The optimal value of the H^∞ norm of the closed-loop transfer function, μ, is the so-called *smallest achievable tolerance.*

It is well known that the first step towards the solution of this H^∞ problem is to replace the contraint "K stabilizing" by a more tractable constraint like "$Q \in H^\infty$," where Q is a parameter that represents the compensator. To be more precise, there exists a one-to-one (linear fractional) mapping from the set of stabilizing compensators K to the space of all $Q \in H^\infty$. Now, reformulating the H^∞ minimization problem in terms of Q rather than K yields, after some manipulation, the so-called *four-block problem*

$$\inf_{Q \in H^\infty} \left\| \begin{matrix} K_{11} - Q & K_{12} \\ K_{21} & K_{22} \end{matrix} \right\|_\infty \qquad (= \mu),$$

where

$$K_{ij} \in L^\infty.$$

This four-block problem is the most general situation that one will encounter. In some cases, depending on the partitioning structure of G conformably with $(w^T u^T)^T$ and $(z^T y^T)^T$, the problem reduces to a more elementary two-block or one-block problem.

It has been shown in several different places [FF], [FD], [JV], [CD] that the smallest achievable tolerance can be explicitly characterized as the norm of a certain operator Γ

$$\mu = \|\Gamma\|.$$

In the four-block case, this operator is defined as

$$\Gamma\colon H^2 \oplus L^2 \to H^{2\perp} \oplus L^2$$
$$\Gamma(f) = P_{H^{2\perp} \oplus L^2}([K_{ij}]f).$$

An approach to evaluating the norm of Γ is to look at the spectrum of $\Gamma^*\Gamma$. But this requires the understanding of the spectral structure of $\Gamma^*\Gamma$. In this paper, we show that $\Gamma^*\Gamma$ is the compact perturbation of a multiplication operator [JS1], from which it follows that $\Gamma^*\Gamma$ has an essential spectrum and in addition some finite eigenvalues, some of them being located beyond the essential spectrum. Therefore, generically, the smallest achievable tolerance is the (square root ot the) largest eigenvalue of $\Gamma^*\Gamma$ or the *largest singular* value of Γ

$$\mu = \sigma_{\max}(\Gamma) = \lambda_{\max}^{1/2}(\Gamma^*\Gamma).$$

The above provides a nice interpretation of the *largest* singular value of Γ, but this has left researchers in a quandary over the interpretation, if any, of the other singular values.

It is the main purpose of this paper to provide an interpretation for *all* singular values of Γ. Let those singular values be listed as

$$(\sup \operatorname{ess\,spec}(\Gamma^*\Gamma))^{1/2} \leqslant \cdots \leqslant \sigma_3 \leqslant \sigma_2 \leqslant \sigma_1 = \mu.$$

Inspiring oneself from [AAK], define $H^\infty(l)$ to be the set of functions of the form $f+g$ where $f \in H^\infty$ and g is rational, antistable with McMillan degree not exceeding l. Then our main result is

$$\inf_{K,\, T_{zw} \in H^\infty(l)} \|T_{zw}\|_\infty = \inf_{Q \in H^\infty(l)} \left\| \begin{matrix} K_{11} - Q & K_{12} \\ K_{21} & K_{22} \end{matrix} \right\|_\infty$$
$$= \sigma_{l+1}.$$

If we look at the L^∞ distance interpretation of σ_{l+1}, the above is nothing other than the *four-block extension of the Adamjam–Arov–Krein problem.* See [GLH] and [FT].

If we rather look at the control theoretic interpretation of the above, it follows that σ_{l+1} is the level of tolerance that can be achieved if the closed-loop system is allowed to have some unstable poles in a number not exceeding l.

One might wonder what is the practical significance of allowing unstable poles in a feedback system. One should bear in mind that the closed-loop stability constraint is imposed upon the *linear model* of the physical system and as such is sometimes irrelevant from a practical standpoint. We are alluding to some high frequency vibration modes in large space structures. If we relieve the constraint of the stability of the high frequency vibration modes *in the linearized model* (in other words, if we allow for some spillover), the *linear model* of the feedback will be unstable. However, the unstable poles won't go far in the right half plane. Indeed, it usually takes a tremendous amount of control effort to move lightly damped, high frequency vibration modes. The reason is that these poles are usually interlaced with nearby zeros so that the root locus goes from the pole to the nearby zero. Putting some weight on the control effort in the H^∞ criterion will prevent the poles from moving too much. Therefore, the worst that could happen in the closed-loop *linear model* is the appearance of some slightly destabilized vibration modes. However, the *real-world* structure, because of nonlinearity, hysteresis, and damping, will not appear unstable. All one will experience in a ground based experiment is a high frequency self-sustained oscillation *noise*, but the amplitude of the motion is so much limited by the nonlinear phenomena that one doesn't see it, nor does it degrade in a substantial manner the pointing accuracy.

As a corollary of this interpretation of the singular values of Γ, it follows that in the rational case one can derive a simple bound on the number of eigenvalues of $\Gamma^*\Gamma$ located beyond the essential spectrum. An a priori bound on the number of eigenvalues of $\Gamma^*\Gamma$ would give an estimate of the complexity of the polynomial algorithm for computing these eigenvalues; see [JJ1, JS2].

Besides the interpretation of the eigenvalues of $\Gamma^*\Gamma$ proposed in the present paper, we mention the work of Zames, Tannenbaum, and Foias [ZTF] dealing with yet another interpretation of the singular values of Γ in case Γ is infinite dimensional Hankel.

The paper is organized as follows: We begin with the mixed sensitivity H^∞ design with l unstable closed-loop poles. This problem is reduced to the four-block Adamjan–Arov–Krein problem which is solved via the Ball–Helton theory. The complete interpretation of *all* singular values of Γ is provided. Finally, we look at the general spectral structure of $\Gamma^*\Gamma$ and derive an upper bound on the number of eigenvalues.

1. Stability of Feedback Systems

Consider the feedback system diagrammed in Fig. 1. To define internal stability for the closed loop system, we introduce auxiliary artificial inputs v_1 and v_2 as indicated in Fig. 2. Stability is defined to mean that the nine transfer matrices from the three inputs w, v_1, v_2 to the signals z, u, y all belong to H^∞; we say in this case that K *stabilizes* G. It turns out (see [FD] or [F]) that if G is stabilizable at all, then to stabilize G is the same

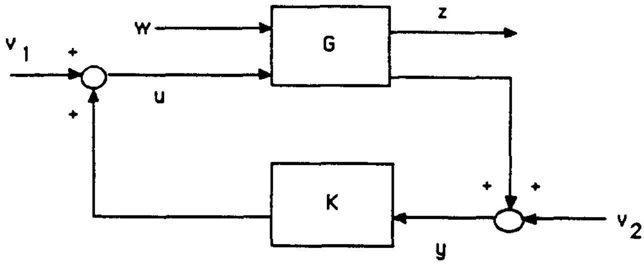

FIGURE 2

as to stabilize G_{22}. We say that K stabilizes G_{22} if, in Fig. 3, the four transfer matrices from v_1, v_2 to u, y are stable. The equations corresponding to Fig. 3 are

$$v_1 + Ky = u$$

$$G_{22}u + v_2 = y,$$

that is,

$$\begin{bmatrix} I & -K \\ G_{22} & -I \end{bmatrix}\begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} v_1 \\ -v_2 \end{bmatrix}. \tag{1.1}$$

Thus stability is equivalent to the condition that $\left[\begin{smallmatrix} I & -K \\ G_{22} & -I \end{smallmatrix}\right]^{-1} \in RH^\infty$. To analyze this, introduce the right coprime factorizations

$$G_{22} = NM^{-1}$$

and

$$K = UV^{-1}$$

for G_{22} and K. Then the transfer matrix from $\left(\begin{smallmatrix} u \\ y \end{smallmatrix}\right)$ to $\left(\begin{smallmatrix} v_1 \\ -v_2 \end{smallmatrix}\right)$ has a factorization

$$\begin{bmatrix} I & -K \\ G_{22} & -I \end{bmatrix} = \begin{bmatrix} M & U \\ N & V \end{bmatrix}\begin{bmatrix} M & O \\ O & -V \end{bmatrix}^{-1}. \tag{1.2}$$

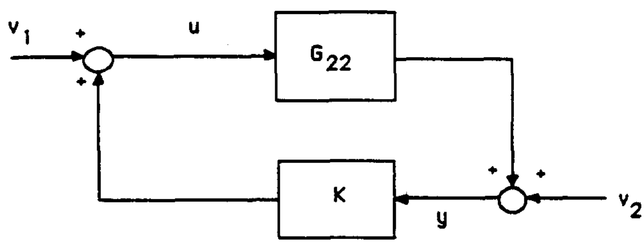

FIGURE 3

It is also convenient to introduce a left coprime factorization $G_{22}=\tilde{M}^{-1}\tilde{N}$ of G_{22} for which there exist matrices $\tilde{X}$, $\tilde{Y}$, X, Y over RH^∞ satisfying the generalized Bezout identity

$$\begin{bmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{bmatrix}\begin{bmatrix} M & Y \\ N & X \end{bmatrix}=I. \tag{1.3}$$

For their existence see [FD] or [F]. Since U, V are right coprime there exist matrices A, B over RH^∞ such that

$$AU+BV=I. \tag{1.4}$$

Identities (1.3) and (1.4) trivially imply

$$\begin{bmatrix} O & -\tilde{Y} \\ A & O \end{bmatrix}\begin{bmatrix} M & U \\ N & V \end{bmatrix}+\begin{bmatrix} \tilde{X} & -\tilde{Y} \\ -A & -B \end{bmatrix}\begin{bmatrix} M & O \\ O & -V \end{bmatrix}=\begin{bmatrix} I & O \\ O & I \end{bmatrix}. \tag{1.5}$$

This shows that (1.2) is a right coprime factorization of $\left[\begin{smallmatrix} I & -K \\ G_{22} & -I \end{smallmatrix}\right]$, and hence $\left[\begin{smallmatrix} I & -K \\ G_{22} & -I \end{smallmatrix}\right]^{-1}$ is in RH^∞ if and only if $\left[\begin{smallmatrix} M & U \\ N & V \end{smallmatrix}\right]$ is a unit over RH^∞. We conclude:

PROPOSITION 1.1. *The compensator $K=UV^{-1}$ (where U, V are right coprime) stabilizes G_{22} if and only if $\left[\begin{smallmatrix} M & U \\ N & V \end{smallmatrix}\right]$ is a unit over RH^∞.*

There is now a standard way for parametrizing all the stabilizing compensators. From (1.3) we see that $\left[\begin{smallmatrix} M & Y \\ N & X \end{smallmatrix}\right]$ is a unit over RH^∞. As is explained in [BH2], any other unit over RH^∞ having first column $\left[\begin{smallmatrix} M \\ N \end{smallmatrix}\right]$ necessarily has the form

$$\begin{aligned}\begin{bmatrix} M & U \\ N & V \end{bmatrix}&=\begin{bmatrix} M & Y \\ N & X \end{bmatrix}\begin{bmatrix} I & -Q \\ O & I \end{bmatrix}\\ &=\begin{bmatrix} M & -MQ+Y \\ N & -NQ+X \end{bmatrix}\end{aligned}$$

for some $Q\in RH^\infty$ of the appropriate size. If $\det(X-NQ)\not\equiv 0$, then by Proposition 1.1, $K=(Y-MQ)(X-NQ)^{-1}$ is a stabilizing compensator. Conversely, the above argument and Proposition 1.1 imply that any such K is of this form for some $Q\in RH^\infty$ with $\det(X-NQ)\not\equiv O$. If we assume that G_{22} is strictly proper, then $N(\infty)=0$ and $\det(X-NQ)\not\equiv O$ for all $Q\in RH^\infty$. We conclude the following.

PROPOSITION 1.2 (see [BH2, F, FD]). *Assume $G_{22}(\infty)=0$. Then the compensator K stabilizes $G_{22}=NM^{-1}$ if and only if*

$$K=(Y-MQ)(X-NQ)^{-1} \tag{1.6}$$

for some $Q \in RH^{\infty}$, where X and Y are as in (1.3).

We next quantify the amount of instability present in the system diagrammed in Fig. 3 if K is as in (1.6) but with $Q \in RH^{\infty}(l)$ rather than in RH^{∞}. Thus suppose Q has a right coprime factorization $Q=ZW^{-1}$ where det W has l zeros (counting multiplicities) in the right half plane. In this case we write $K=(Y-MQ)(X-NQ)^{-1}$ as

$$K=(YW-MZ)(XW-NZ)^{-1}. \tag{1.7}$$

From the identity

$$\begin{bmatrix} YW-MZ \\ XW-NZ \end{bmatrix}=\begin{bmatrix} M & Y \\ N & X \end{bmatrix}\begin{bmatrix} -Z \\ W \end{bmatrix}$$

and the fact that $\left[\begin{smallmatrix} M & Y \\ N & X \end{smallmatrix}\right]$ is a unit in RH^{∞}, we see that (1.7) is a right coprime factorization for K whenever $Q=ZW^{-1}$ is a right coprime factorization of Q. Thus, setting $U=YW-MZ$ and $V=XW-NZ$ we get that

$$\begin{bmatrix} M & U \\ N & V \end{bmatrix}=\begin{bmatrix} M & Y \\ N & X \end{bmatrix}\begin{bmatrix} I & -Z \\ O & W \end{bmatrix}.$$

Thus from (1.2) the inverse of the transfer matrix from $\left(\begin{smallmatrix} v_1 \\ -v_2 \end{smallmatrix}\right)$ to $\left(\begin{smallmatrix} u \\ y \end{smallmatrix}\right)$ has right coprime factorization

$$\begin{bmatrix} I & -K \\ G_{22} & -I \end{bmatrix}=\begin{bmatrix} M & Y \\ N & X \end{bmatrix}\begin{bmatrix} I & -Z \\ O & W \end{bmatrix}\cdot\begin{bmatrix} M & O \\ O & NZ-XW \end{bmatrix}^{-1}.$$

Since this is a coprime factorization we see that the poles in the right half plane of $\left[\begin{smallmatrix} I & -K \\ G_{22} & -I \end{smallmatrix}\right]^{-1}$ are precisely the same as the poles in the right half plane of $\left[\left[\begin{smallmatrix} M & Y \\ N & X \end{smallmatrix}\right]\left[\begin{smallmatrix} I & -Z \\ O & W \end{smallmatrix}\right]\right]^{-1}$. As $\left[\begin{smallmatrix} M & Y \\ N & X \end{smallmatrix}\right]$ is a unit in RH^{∞}, the number of poles (counting multiplicites) in the right half plane are the same as for $\left[\begin{smallmatrix} I & -Z \\ O & W \end{smallmatrix}\right]^{-1}=\left[\begin{smallmatrix} I & Z \\ O & I \end{smallmatrix}\right]\left[\begin{smallmatrix} I & O \\ O & W \end{smallmatrix}\right]^{-1}$. As $W \in RH^{\infty}$ and det W has l zeros in the right half plane, we conclude that the number of poles in the right half plane of $\left[\begin{smallmatrix} I & -K \\ G_{22} & -I \end{smallmatrix}\right]^{-1}$ is precisely l. Thus we have:

PROPOSITION 1.3. *Assume $G_{22}(\infty)=0$. The compensator K gives rise to a* *transfer matrix* T: $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \to \begin{bmatrix} u \\ y \end{bmatrix}$ *having* l *poles* (*counting multiplicity*) *in the right half plane if and only if* K *is given by formula* (1.6)

$$K = (Y - MQ)(X - NQ)^{-1},$$

where $Q \in RH^{\infty}(l)$.

Proof. It remains only to show the "only if" part. This can be obtained by reversing the above argument; each step actually was a necessary and sufficient analysis. ∎

Remark. Observe from the above analysis that the unstable poles of Q are also the unstable poles of the closed-loop system. This would allow for some further control of the closed-loop unstable poles which are required to remain not too far from the imaginary axis. This could be handled using conformal mapping techniques.

2. The H^{∞}-Optimality Problem

In the terminology of Francis and Doyle ([FD] or [F]) the *standard problem* pertains to Fig. 1. It is assumed that G is real rational and proper (analytic at $s = \infty$), and is partitioned as

$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$$

so the equations corresponding to Fig. 1 are

$$z = G_{11} w + G_{12} u$$

$$y = G_{21} w + G_{22} u$$

$$u = Ky.$$

When one eliminates u and y, one gets the transfer matrix from w to z to be a linear fractional transformation of K:

$$z = [G_{11} + G_{12} K(I - G_{22} K)^{-1} G_{21}] w. \tag{2.1}$$

Note that with our standing assumption that $G_{22}(\infty) = 0$ the inverse always exists as a rational matrix function. In the terminology of [FD], the *standard problem* is: find a real rational proper K to minimize the H^{∞}-norm of the transfer function from w to z under the constraint that K stabilizes G (or G_{22}). We propose here to consider the *modified standard problem*: find K to minimize $\|G_{11} + G_{12} K(I - G_{22} K)^{-1} G_{21}\|_{\infty}$ under the constraint that the closed loop transfer function $\left[\begin{smallmatrix} I & -K \\ G_{22} & -I \end{smallmatrix}\right]^{-1}$ from $\left[\begin{smallmatrix} v_1 \\ -v_2 \end{smallmatrix}\right]$ to $\left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]$ in Fig. 3 has at most l unstable poles (including multiplicities).

The problem assumes a more affine form if we use the parameter $Q \in RH^{\infty}(l)$ given by Proposition 1.3 rather than the compensator. We need only plug in formula (1.6) for K in the expression $G_{11} + G_{12}K(I - G_{22}K)^{-1} G_{21}$ and simplify algebraically to obtain the following.

PROPOSITION 2.1. *With K as in* (1.6) *the transfer matrix from w to z equals $T_1 - T_2QT_3$ where $T_j \in RH^{\infty}$ are given by*

$$T_1 = G_{11} + G_{12} Y\tilde{M} G_{21} \tag{2.2}$$

$$T_2 = G_{12} M \tag{2.3}$$

$$T_3 = \tilde{M} G_{21}. \tag{2.4}$$

Thus the modified standard problem is reduced to find $Q \in RH^{\infty}(l)$ to minimize $\|T_1 - T_2QT_3\|_{\infty}$.

It is convenient to reduce the problem one step further. We *assume* that $T_2(j\omega)$ and $T_3(j\omega)$ have constant ranks. Without loss of generality we may assume that T_2 has at least as many rows as columns ("tall" matrix size) while T_3 has the reverse property of at least as many columns as rows ("fat" matrix size). We introduce the inner-outer factorization

$$T_2 = (T_2)_i \, (T_2)_o \tag{2.5}$$

of T_2 and the co-inner/co-outer factorization

$$T_3 = (T_3)_{co} \, (T_3)_{ci} \tag{2.6}$$

of T_3 as in [FD]. Then $(T_2)_o$ is right invertible over RH^{∞} and $(T_3)_{co}$ is left-invertible over RH^{∞}. Thus the mapping

$$Q \to (T_2)_o \, Q (T_3)_{co}$$

is a surjective mapping of $RH^{\infty}(l)$ to itself. Thus a problem equivalent to the modified standard problem is

$$\min_{Q \in RH^{\infty}(l)} \|T_1 - (T_2)_i \, Q (T_3)_{ci}\|_{\infty}.$$

Let us suppose T_1 has size $m \times n$, $(T_2)_i$ has size $m \times m_1$, and $(T_3)_{ci}$ has size $n_1 \times n$. Introduce the $(m + m_1) \times m$ matrix

$$E := \begin{bmatrix} (T_2)_i^* \\ I - (T_2)_i \, (T_2)_i^* \end{bmatrix} \tag{2.7}$$

(where $W^*(s) = W(-\bar{s})^*$). Then since $(T_2)_i$ is inner, we get $E^*E = I_m$. Similarly, $L^*L = I_n$, where L is the $(n_1+n)\times n$ matrix function

$$L := \begin{bmatrix} (T_3)_{ci} \\ I - (T_3)_{ci}^* (T_3)_{ci} \end{bmatrix}. \tag{2.8}$$

Then as in [FD],

$$\|T_1 - (T_2)_i Q (T_3)_{ci}\|_\infty = \left\| \begin{bmatrix} K_{11} - Q & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \right\|_\infty,$$

where

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} = ET_1 L^*. \tag{2.9}$$

Here K_{ij} has size $m_i \times n_j$ (where $m_2 = m$, $n_2 = n$). Thus the final reduction is:

PROPOSITION 2.2. *The modified standard problem is equivalent to: find* $Q \in RH^\infty(l)$ *to minimize the* ∞*-norm of* $\left[\begin{smallmatrix} K_{11}-Q & K_{12} \\ K_{21} & K_{22} \end{smallmatrix}\right]$ *where* K_{ij} *is given by* (2.2)–(2.9).

3. The Grassmannian Approach

The following theorem is basic to our analysis of the modified standard problem as formulated in Proposition 2.2. For the case $m_2 = n_2 = 0$ it can be found in [BH1]; the case $l = 0$ is presented in [BC].

THEOREM 3.1. *Let rational matrix functions* $K_{ij} \in RL^\infty$ *of sizes* $m_i \times n_j$ $(j = 1, 2)$ *be given, and let a tolerance level* $\mu > 0$ *be given. Then a subspace* $\mathcal{G} \subset L_M^2 \oplus H_{n_1}^2 \oplus L_{n_2}^2$ *is of the form*

$$\mathcal{G} = \begin{bmatrix} F \\ I_N \end{bmatrix} \begin{bmatrix} \psi H_{n_1}^2 \\ L_{n_2}^2 \end{bmatrix},$$

where

(1)

$$F = \begin{bmatrix} K_{11} - Q\psi^{-1} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}$$

for some $n_1 \times n_1$ matrix Blaschke product ψ of degree l and some $Q \in RH^\infty$ of size $m_1 \times n_1$

(2) $\|F\|_\infty \leqslant \mu$

if and only if

(i) *$\mathscr{G}$ is J_μ-negative and has codimension l in a maximal J_μ-negative subspace in $L_M^2 \oplus H_{n_1}^2 \oplus L_{n_2}^2$. Here*

$$J_\mu = \begin{bmatrix} I_M & O \\ O & -\mu^2 I_N \end{bmatrix}.$$

(ii) *$\mathscr{G}$ is a subspace of the subspace*

$$\mathscr{M} = L(H_{m_1}^2 \oplus H_{n_1}^2 \oplus L_{n_2}^2),$$

where

$$L = \begin{bmatrix} I_{m_1} & K_{11} & K_{12} \\ 0 & K_{21} & K_{22} \\ 0 & I_{n_1} & 0 \\ 0 & 0 & I_{n_2} \end{bmatrix}.$$

(iii) *$\mathscr{G}$ is shift invariant, that is, $\mathscr{X}\mathscr{G} \subset \mathscr{G}$, where $\mathscr{X}(s) = (s-1)/(s+1)$.*

Proof. The proof is a straightforward adaptation of the analysis for the special cases in [BH1, BC]. A key point is that a subspace $\mathscr{N}$ of $H_{n_1}^2 \oplus L_{n_2}^2$ is shift invariant and simultaneously of codimension l in $H_{n_1}^2 \oplus L_{n_2}^2$ if and only if $\mathscr{N} = \psi H_{n_1}^2 \oplus L_{n_2}^2$ for a $n_1 \times n_1$ matrix Blaschke product of degree l. ∎

The next immediate question is to characterize when conditions (i) and (ii) are compatible, i.e., when do solutions $\mathscr{G}$ of conditions (i) and (ii) only exist. This is answered by the next lemma.

LEMMA 3.2. *A subspace $\mathscr{G}$ of $\mathscr{M}$ which is maximal J_μ-negative in $\mathscr{M}$ has codimension l in a subspace $\tilde{\mathscr{G}}$ which is maximal J_μ-negative in $L_M^2 \oplus H_{n_1}^2 \oplus L_{n_2}^2$ if and only if the subspace $\mathscr{S} = [L_M^2 \oplus H_{n_1}^2 \oplus L_{n_2}^2] \cap \mathscr{M}^{\perp J_\mu}$ has l negative squares. Equivalently, if*

$$\Gamma \colon H_{n_1}^2 \oplus L_{n_2}^2 \to H_{m_1}^{2\perp} \oplus L_{m_2}^2$$

is defined by

$$\Gamma(f) = P_{H_{m_1}^{2\perp} \oplus L_{m_2}^2}(Kf),$$

then the rank of the spectral projection $P(\Gamma^*\Gamma; (\mu^2, \infty))$ *of* $\Gamma^*\Gamma$ *for the interval* (μ^2, ∞) *is* l.

Proof. The first assertion follows as in Lemma 1.1 of [BH1]. To verify the second assertion we need to compute $\mathcal{S}$ more explicitly. One can check that $\mathcal{S}$ has the form

$$\mathcal{S} = \{f \oplus \mu^{-2}\Gamma^*(f): f \in H^{2\perp}_{m_1} \oplus L^2_{m_2}\}.$$

Thus the J_μ-inner product on $\mathcal{S}$ is congruent to the inner product on $H^2_{n_1} \oplus L^2_{n_2}$ induced by the Hermitian operator

$$H := [I, \mu^{-2}\Gamma]\, J_\mu \begin{bmatrix} I \\ \mu^{-2}\Gamma^* \end{bmatrix} = I - \mu^{-2}\Gamma\Gamma^*.$$

Thus the number of negative squares had by $\mathcal{S}$ in the J_μ-inner product is the same as the number of negative squares had by $H^2_{n_1} \oplus L^2_{n_2}$ in the H-inner product. This clearly is the same as the rank of the spectral projector of $\Gamma^*\Gamma$ for the interval (μ^2, ∞). ∎

By Lemma 3.2, a necessary and sufficient condition for the existence of a subspace $\mathcal{G}$ satisfying conditions (i) and (ii) in Theorem 3.1 is that rank $P(\Gamma^*\Gamma; (\mu^2, \infty)) \leqslant l$. Thus this condition is certainly necessary for the existence of subspaces $\mathcal{G}$ satisfying (i), (ii), and (iii). From this we get the inequality

$$\begin{aligned} &\inf_{Q \in H^\infty(l)} \left\| \begin{bmatrix} K_{11} - Q & K_{12} \\ K_{22} & K_{22} \end{bmatrix} \right\|_\infty \\ &\qquad \geqslant \inf\{\mu: \text{rank } P(\Gamma^*\Gamma; (\mu^2, \infty)) \leqslant l\}. \end{aligned} \tag{3.1}$$

This inequality can also easily be seen directly without using the Grassmannian machinery.

[Indeed for each $Q \in H^\infty(l)$, the infinity norm of $\left[\begin{smallmatrix} K_{11} - Q & K_{12} \\ K_{21} & K_{22} \end{smallmatrix}\right]$ is the same as its operator norm as a multiplication operator on $H^2_{n_1} \oplus L^2_{n_2}$. If $Q \in H^\infty(l)$ then the operator $f \in H^2_{n_1} \to P_{H^{2\perp}_{m_1}}(Qf) \in H^{2\perp}_{m_1}$ has rank l. Thus

$$\begin{aligned} \left\| \begin{bmatrix} K_{11} - Q & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \right\| &\geqslant \left\| P_{H^{2\perp}_{m_1} \oplus L^2_{m_2}} \begin{bmatrix} K_{11} - Q & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \right\| \\ &= \left\| \Gamma - \begin{bmatrix} P_{H^{2\perp}_{m_1}} Q \mid H^2_{n_1} & 0 \\ 0 & 0 \end{bmatrix} \right\| \\ &\geqslant \inf\{\|\Gamma + X\|: \text{rank } X \leqslant l\} \\ &\geqslant \inf\{\mu: \text{rank } P(\Gamma^*\Gamma; (\mu^2, \infty)) \leqslant l\}, \end{aligned}$$

where the last step follows from the singular value decomposition.]

To reverse the inequality in (3.1), we need to show that there always exist maximal J_μ-negative subspaces in $\mathcal{M}$ which are also shift invariant. To accomplish this we need a Beurling–Lax type representation theorem for the subspace $\mathcal{M}$ which was proved in [BC].

THEOREM 3.3. *Assume $\Gamma^*\Gamma-\mu^2 I$ is invertible. Then there exists a rational $(M+N)\times(m_1+n_1+n_2)$ matrix function*

$$\theta=\begin{bmatrix}\theta_{11} & \theta_{12} & \theta_{13}\\ \theta_{21} & \theta_{22} & \theta_{23}\end{bmatrix}$$

such that

(i) $\mathcal{M}=\theta\cdot[H^2_{m_1}\oplus H^2_{n_1}\oplus L^2_{n_2}]$ *and*

(ii) $\theta^* J_\mu \theta = J$,

where $J=I_{m_1}\oplus -I_N$.

With the assumption that $\Gamma^*\Gamma-\mu^2 I$ is invertible, Theorem 3.3 enables us to describe all the subspaces which are maximal J_μ-negative in $\mathcal{M}$ and simultaneously shift invariant.

PROPOSITION 3.4. (see [BC]). *Assume $\Gamma^*\Gamma-\mu^2 I$ is invertible and that θ is as in Theorem 3.3. Then a subspace $\mathcal{G}$ is maximal J_μ-negative in $\mathcal{M}$ and shift invariant if and only if*

$$\mathcal{G}=\theta\mathcal{G}_1,$$

where $\mathcal{G}_1$ is maximal J-negative in $H^2_{m_1}\oplus H^2_{n_1}\oplus L^2_{n_1}$ and shift invariant, that is, where

$$\mathcal{G}_1=\begin{bmatrix}G & 0\\ I & O\\ O & I\end{bmatrix}\begin{bmatrix}H^2_{n_1}\\ \\ L^2_{n_2}\end{bmatrix}$$

for some G in $H^\infty_{m_1\times n_1}$ with $\|G\|_\infty\leqslant 1$.

When we put all the pieces together we get the following extension of one of the main results from [BC].

THEOREM 3.5. *Suppose $\mu^2 I-\Gamma^*\Gamma$ is invertible and* rank $P(\Gamma\Gamma;(\mu^2,\infty))$ $=l$. *Let θ be as in Theorem* 3.3. *Then the formula*

$$F=[\theta_{11}G+\theta_{12},\theta_{13}][\theta_{21}G+\theta_{22},\theta_{23}]^{-1}$$

gives a one-to-one correspondence between rational $M_1 \times N_1$ matrix functions G over RH^∞ with $\|G\|_\infty \leqslant 1$ and rational $M \times N$ matrix functions F of the form

$$F = \begin{bmatrix} K_{11} - Q & K_{12} \\ K_{21} & K_{22} \end{bmatrix}$$

with $Q \in RH^\infty(l)$ of size $m_1 \times n_1$ such that

$$\|F\|_\infty \leqslant \mu.$$

COROLLARY 3.6.

$$\inf_{Q \in RH^\infty(l)} \left\| \begin{bmatrix} K_{11} - Q & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \right\|_\infty$$
$$= \inf\{\mu \colon P(\Gamma^*\Gamma; (\mu^2, \infty)) \leqslant l\}.$$

Corollary 3.6 was conjectured by Jonckheere and Verma [JV] for the special case $n_2 = 0$.

4. SPECTRAL PROPERTIES OF THE $\Gamma^*\Gamma$ OPERATOR

The central object of concern in the problem of evaluating the achievable performance under the condition that the closed-loop system has no more than l unstable poles is the spectrum of the $\Gamma^*\Gamma$ operator. This was precisely stated by Corollary 3.6. In this section, we analyze the spectral properties of $\Gamma^*\Gamma$. We show that $\Gamma^*\Gamma$ has, in addition to a continuous spectrum embedded in $\mathbb{R}^+$, some finite multiplicity eigenvalues located on the real line to the right of the continuous spectrum. The finite multiplicity eigenvalues, rather than the continuous spectrum, determine the level of tolerance that can be reached by a feedback system with no more than l unstable closed-loop poles.

4.1. *The Particular Case $n_2 = 0$*

For clarity of the exposition, we begin with the simple case $n_2 = 0$, i.e.,

$$\inf_{Q \in RH^\infty(l)} \left\| \begin{matrix} K_{11} - Q \\ K_{21} \end{matrix} \right\|_\infty, \tag{4.1}$$

where $K_{ij} \in RL^\infty$.

It is easily seen that the infinity norm depends on the product $K_{21}^* K_{21}$

rather than K_{21} itself and therefore by spectral factorization there is no loss of generality in assuming that

$$K_{21} \in RH^{\infty}.$$

In this situation, the Γ-operator becomes

$$\Gamma = \begin{bmatrix} P^{\perp}K_{11} \\ PK_{21} \end{bmatrix} : H^2_{n_1} \to H^{2\perp}_{m_1} \oplus H^2_{m_2}.$$

Clearly,

$$H := P^{\perp}K_{11} : H^2_{n_1} \to H^{2\perp}_{m_1}$$

is *Hankel*, while

$$T := PK_{21} : H^2_{n_1} \to H^2_{m_2}$$

is *Toeplitz*. Therefore, the $\Gamma^*\Gamma$ operator becomes

$$\Gamma^*\Gamma = H^*H + T^*T : H^2_{n_1} \to H^2_{n_1}.$$

This is the so-called "*Toeplitz + Hankel*" operator.

The "Toeplitz + Hankel" operator structure within the context of system theory emerged in a series of papers by Jonckheere and Silverman [JS1–JS3] dealing with the Linear-Quadratic (LQ) problem. Later, the same "Toeplitz + Hankel" operator structure was found to play a crucial role in the H^{∞} problem—see Francis and Feintuch [FF], Jonckheere and Verma [JV], Verma and Jonckheere [VJ], and Chu and Doyle [CD]. The common "Toeplitz + Hankel" operator structure shared by the seemingly unrelated Linear-Quadratic and H^{∞} problems emerged as a tool for fast computation of achievable H^{∞} performance in work by Jonckheere and Juang [JJ2].

We now develop the spectral theory of the "Toeplitz + Hankel" operator. The following two lemmas are well known and easily proved.

LEMMA 4.1. *H^*H is compact.*

LEMMA 4.2.

$$\begin{aligned}&\text{ess spec}(T^*T + H^*H)\\ &\qquad = \text{ess spec}(T^*T) = \text{closure}\{\lambda(K^T_{21}(-j\omega)\, K_{21}(j\omega)) : \omega \in \mathbb{R}\}.\end{aligned}$$

From the above, it follows that the essential spectrum of $T^*T + H^*H$ is a compact subset of the real line—the locus of the eigenvalues of

$K_{21}^T(-j\omega)\,K_{21}(j\omega)$ as ω goes from $-\infty$ to $+\infty$. Moreover, since a rational matrix function with positive values on the $j\omega$ axis has a spectral factorization, it is easy to see that spec(T^*T) is contained in the convex hull of ess spec(T^*T).

In addition to the essential spectrum, the operator T^*T+H^*H has, in general, some eigenvalues with finite dimensional eigenspaces. Where these eigenvalues are located relative to the essential spectrum and how many of them are present are questions that have been around ever since the original paper of Jonckheere and Silverman [JS1].

Since $\|T^*T+H^*H\| \geqslant \|T^*T\|$ and since the essential part of the spectrum of T^*T+H^*H does not extend beyond $\|T^*T\| = \|K_{21}\|_\infty^2$, there are, in general, some finite multiplicity eigenvalues of T^*T+H^*H beyond $\|K_{21}\|_\infty^2$.

We now state the following precise result as to how many eigenvalues are located up there.

THEOREM 4.1. *The number (multiplicity counted) of eigenvalues of T^*T+H^*H located within $(\|K_{21}\|_\infty^2, \infty)$ does not exceed the number of unstable poles of K_{11}.*

Proof. Consider the problem (4.1) without any stability constraint. Clearly,

$$\inf_{Q\in RL^\infty} \left\| \begin{matrix} K_{11}-Q \\ K_{21} \end{matrix} \right\|_\infty = \|K_{21}\|_\infty$$

and the infimum is achieved for

$$Q = K_{11}.$$

Let d be the degree of the unstable part of K_{11}. Hence

$$\inf_{Q\in RL^\infty} \left\| \begin{matrix} K_{11}-Q \\ K_{21} \end{matrix} \right\|_\infty = \inf_{Q\in RH^\infty(d)} \left\| \begin{matrix} K_{11}-Q \\ K_{21} \end{matrix} \right\|_\infty = \|K_{21}\|_\infty.$$

Now from Corollary 3.6,

$$\|K_{21}\|_\infty = \inf\{\mu\colon \text{rank } P(T^*T+H^*H;\,(\mu^2,\ \infty)) \leqslant d\}.$$

Clearly, it follows that the rank of the spectral projection of T^*T+H^*H on $(\|K_{21}\|_\infty^2, \infty)$ does not exceed d. ∎

The above result was conjectured for the first time in Jonckheere and Silverman [JS1] in the seemingly unrelated linear-quadratic context. Recently, two proofs have been constructed: the proof of Jonckheere and

Verma [JV] based on perturbation theory and the proof of Juang and Jonckheere [JJ1] based on a polynomial formulation of the eigenvalue problem. The above proof is, however, by far the most economical.

Now, we look at what's going on below the infimum of the essential spectral. We show that there are no eigenvalues down there.

THEOREM 4.2. *There are no finite multiplicity eigenvalues below the infimum of the essential spectrum.*

Proof. By contradiction, assume there exists a finite multiplicity eigenvalue λ of T^*T+H^*H below the essential spectrum:

$$\begin{aligned}\lambda &< \inf \operatorname{ess\,spec}(T^*T+H^*H)\\ &= \inf\{\lambda_{\min}(K_{21}^T(-j\omega)\,K_{21}(j\omega))\colon \omega\in\mathbb{R}\}.\end{aligned} \tag{4.2}$$

Clearly,

$$\begin{aligned}&\frac{\|(T^*T+H^*H)v\|}{\|v\|}\\ &\geqslant \frac{\|T^*Tv\|}{\|v\|}\geqslant \inf_w \frac{\|T^*Tw\|}{\|w\|}=\inf \operatorname{spec}(T^*T)\\ &=\inf \operatorname{ess\,spec}(T^*T)=\inf \operatorname{ess\,spec}(T^*T+H^*H) \qquad \forall v\in H^2_{n_1}.\end{aligned}$$

Now taking v to be the eigenvector of T^*T+H^*H associated with the eigenvalue λ yields

$$\lambda \geqslant \inf \operatorname{ess\,spec}(T^*T+H^*H). \tag{4.3}$$

Inequalities (4.2) and (4.3) are clearly contradicting each other. ∎

4.2. *The General Case $n_2\neq 0$*

The road to the elucidation of the spectral properties of the operator $\Gamma^*\Gamma$ in the general four-block case is, as in the two-block case, the decomposition of $\Gamma^*\Gamma$ as a "Toeplitz" operator plus a compact Hankel-like perturbation.

It is a matter of trivial manipulation to deduce

$$\begin{aligned}\Gamma^*\Gamma &= \begin{bmatrix} PK_{11}^*P^\perp K_{11}P+PK_{21}^*K_{21}P & PK_{11}^*P^\perp K_{12}+PK_{21}^*K_{22}\\ K_{12}^*P^\perp K_{11}P+K_{22}^*K_{21}P & K_{12}^*P^\perp K_{12}+K_{22}^*K_{22}\end{bmatrix}\\ &\quad : H^2_{n_1}\oplus L^2_{n_2}\to H^2_{n_1}\oplus L^2_{n_2}.\end{aligned}$$

Clearly, we have the liberty of postmultiply the $(1,2)$ block by $P+P^\perp$ $(=I)$, to premultiply the $(2,1)$ block by $P+P^\perp$, and finally to premultiply and postmultiply the (2, 2) block by $P+P^{\perp}$. After some manipulations this yields

$$\Gamma^*\Gamma=\begin{pmatrix} T_{11} & T_{12} \\ T_{12}^* & T_{22} \end{pmatrix}+\begin{pmatrix} H_{11} & H_{12} \\ H_{12}^* & H_{22} \end{pmatrix},$$

where

$$\begin{aligned}
T_{11} &:= PK_{21}^*K_{21}P \\
T_{12} &:= PK_{21}^*K_{22}P \\
T_{22} &:= PK_{22}^*K_{22}P+P^{\perp}(K_{22}^*K_{22}+K_{12}^*K_{12})\,P^{\perp} \\
H_{11} &:= PK_{11}^*P^{\perp}K_{11}P \\
H_{12} &:= PK_{11}^*P^{\perp}K_{12}+PK_{21}^*K_{22}P^{\perp} \\
H_{22} &:= P^{\perp}K_{12}^*P^{\perp}K_{12}P+PK_{12}^*P^{\perp}K_{12}P \\
&\quad +PK_{12}^*P^{\perp}K_{12}P^{\perp}+PK_{22}^*K_{22}P^{\perp} \\
&\quad +P^{\perp}K_{22}^*K_{22}P-P^{\perp}K_{12}^*PK_{12}P^{\perp}.
\end{aligned}$$

The following lemmas constitute the key.

LEMMA 4.3. *The operator H is compact.*

Proof. The (1, 1) block of H, H_{11}, is $(PK_{11}^*P^{\perp})\,K_{11}P$. Clearly, $(PK_{11}^*P^{\perp})$ is compact Hankel. Hence $(PK_{11}^*P^{\perp})\,K_{11}P$ is compact because it is the product of a compact operator and a bounded operator. The same argument applies to the other blocks. Hence H is compact. ∎

LEMMA 4.4. *The operator $T=\left(\begin{smallmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{smallmatrix}\right)$ is unitarily equivalent to*

$$T':=\begin{bmatrix} PK_{21}^*K_{21}P & PK_{21}^*K_{22}P & 0 \\ PK_{22}^*K_{21}P & PK_{22}^*K_{22}P & 0 \\ 0 & 0 & P^{\perp}(K_{22}^*K_{22}+K_{12}^*K_{12})\,P^{\perp} \end{bmatrix}.$$

To be more precise, the following diagram commutes:

$$\begin{array}{ccc}
H_{n_1}^2\oplus L_{n_2}^2 & \xrightarrow{\quad T\quad} & H_{n_1}^2\oplus L_{n_2}^2 \\
\updownarrow & & \updownarrow \\
H_{n_1}^2\oplus H_{n_2}^2\oplus H_{n_2}^{2\perp} & \xrightarrow{\quad T'\quad} & H_{n_1}^2\oplus H_{n_2}^2\oplus H_{n_2}^{2\perp}
\end{array}$$

Proof. The fact that T and T' are unitarily equivalent should be clear from the above commutative diagram. ∎

Because T and T' are unitarily equivalent, they have the same spectrum. Further, since T' is the direct sum of the Toeplitz operator $P\binom{K_{21}^*}{K_{22}^*}(K_{21}\ K_{21})\,P$ and the (reverse) Toeplitz operator $P^{\perp}(K_{22}^* K_{22} + K_{12}^* K_{12})\,P^{\perp}$, the following easily emerges:

LEMMA 4.5.

$$\begin{aligned}\text{ess spec}(T) &= \text{ess spec}(T') \\ &= \left\{\lambda \begin{bmatrix} K_{21}^T(-j\omega) \\ K_{22}^T(-j\omega) \end{bmatrix} [K_{21}(j\omega)\ \ K_{22}(j\omega)] : \omega \in \mathbb{R}\right\} \\ &\quad \cup \{\lambda(K_{22}^T(-j\omega)\,K_{22}(j\omega) + K_{12}^T(-j\omega)\,K_{12}(j\omega)) : \omega \in \mathbb{R}\}.\end{aligned}$$

Further, from classical perturbation theory, we have the following:

LEMMA 4.6. $\text{ess spec}(\Gamma^*\Gamma) = \text{ess spec}(T)$.

To summarize the situation, the essential spectrum of the 2×2 block operator $\Gamma^*\Gamma$ consists of the locis of $\sigma_i^2(K_{21}(j\omega), K_{22}(j\omega))$ and $\sigma_i^2\left[\begin{smallmatrix} K_{12}(j\omega) \\ K_{22}(j\omega) \end{smallmatrix}\right]$ as ω goes from 0 to ∞. In addition to this essential spectrum the operator $\Gamma^*\Gamma$ has some finite multiplicity eigenvalues. We are of course most interested with those eigenvalues of $\Gamma^*\Gamma$ that occur beyond $\max\{\|K_{21}\ \ K_{22}\|_\infty^2, \|\begin{smallmatrix} K_{12} \\ K_{22} \end{smallmatrix}\|_\infty^2\}$.

We now generalize Theorem 4.1 to the four-block case by providing an upper bound on the number of eigenvalues of $\Gamma^*\Gamma$ occurring beyond the supremum of its essential spectrum.

THEOREM 4.3. *Consider the modified four-block problem of Proposition* 2.2 *subject to the restriction*

$$\|K_{22}\|_\infty < \left\| \begin{matrix} K_{22} \\ K_{12} \end{matrix} \right\|_\infty \leqslant \|K_{21}\ \ K_{22}\|_\infty =: \gamma$$

(*only the first inequality is a restriction*). *Then the number of finite multiplicity eigenvalues of $\Gamma^*\Gamma$ located beyond the supremum of the essential spectrum does not exceed the number of unstable poles of*

$$K_{11} + K_{12}(\gamma^2 I - K_{22}^* K_{22})^{-1} K_{22}^* K_{21}.$$

Proof. The argument is essentially the same as that of Theorem 4.1. They key idea is to consider the modified 2×2 block problem with all stability restrictions relaxed:

$$\inf_{Q \in RL^\infty} \left\| \begin{matrix} K_{11} - Q & K_{12} \\ K_{21} & K_{22} \end{matrix} \right\|_\infty.$$

The solution of this problem is provided in Parrot [P], as well as in Davis, Kahane, and Weinberger [DKW]. From Corollary 1.2 of this paper, it follows that the optimal solution is

$$K_{11} - Q = -K_{12}(\gamma^2 I - K_{22}^* K_{22})^{-1} K_{22}^* K_{21}$$

yielding the optimal cost γ. Now, the same argument as Theorem 4.1 yields the result. ∎

5. Conclusions

As shown in this paper, the complete interpretation of *all* singular values of the four-block operator Γ has required some relaxation of the stability constraint that is usually imposed upon the closed loop system in mixed sensitivity H^∞ optimization. To be more precise, the eigenvalues of $\Gamma^*\Gamma$ occurring beyond the supremum of the essential spectrum are the various levels of tolerance that can be achieved if we allow a various number of unstable poles in the closed loop system. This simple, transparent feedback interpretation has allowed the construction of a simple proof of the fact that the number of those eigenvalues is related, in a simple manner, to the degree of the symbol. This result has been "targeted" ever since the paper of Jonckheere and Silverman [JS1].

Are there eigenvalues between the compact intervals of the essential spectrum? Are there eigenvalues embedded in the continuous spectrum? These are questions which are, to the best of our knowledge, open.

But probably the most outstanding problem is to reinterpret the results of the present paper in the linear-quadratic context. Indeed, as shown in Jonckheere and Silverman [JS1–JS3], the two-block "Toeplitz + Hankel" operator $\Gamma^*\Gamma$ also plays a central role in the linear quadratic problem.

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